Differential and integral calculus

Frederic Trillaud

National Autonomous University of Mexico

ftrillaudp@pumas.ii.unam.mx

February 26, 2016

Introduction

The objective is twofold:

- Review the basic material related to differential and integral calculus
- Give the basis to prepare the exam

It is up to you to get prepared, so review your notes from your undergraduate classes and the guide of applied mathematics provided by the graduate school of electrical engineering of the UNAM:
http://posgrado.electrica.unam.mx/pagina/guiasdeexamenes

Differential and integral calculus: an introduction

Differential calculus

Differential calculus is concerned about rate of change (slopes).

Integral calculus

Integral calculus is concerned about total (sums).

Knowing the rate of change of a function or knowing its sum at any value is ”equivalent” (fundamental theorem of calculus).

Functions

Definition

A function is a relation between an input and an output where every input gets a single unique output

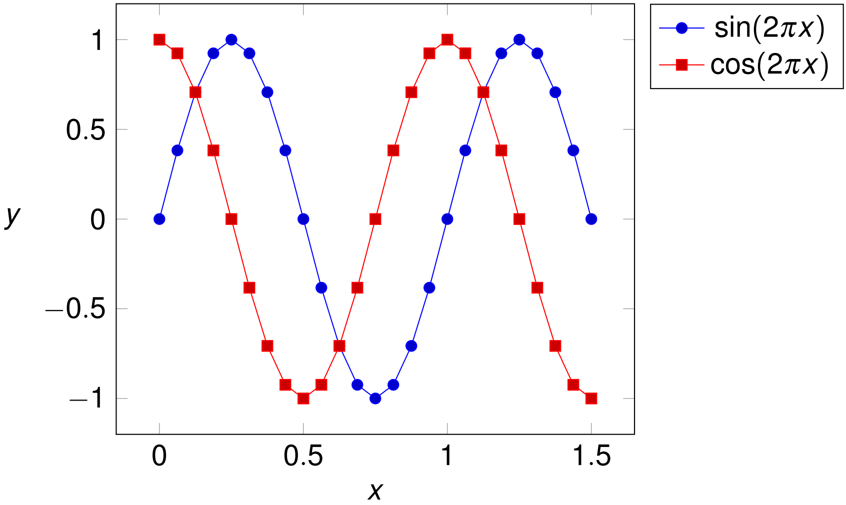
What do we do with a function?

A good practice is to plot a function for which we need to know the domain of definition and some properties.

An example of an elementary function:

$$f(x) = \sin(x), \; \forall x \in \mathbb{R}$$

Functions - example



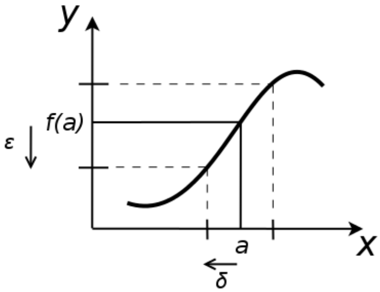
Limits

Definition of a limit in $\mathbb{R}$

$$\lim_{x \to a} f(x) = L$$

$\forall \varepsilon > 0$, there exists a number $\delta > 0$ so that if $|x - a| < \delta$, then $|f(x) - L| < \varepsilon$.

Geometric interpretation:



Limits - indeterminate form

The following forms are indeterminate meaning that there is not enough information to make a decision:

$\dfrac{0}{0}$, $\dfrac{\infty}{\infty}$, $\infty \times 0, \infty - \infty$, 0^0, 1^∞, and ∞^0

Examples:

$$\lim_{x \to 0^+} 0^x = 0, \;\; \lim_{x \to 0} x^0 = 1$$

$$\lim_{x \to 4} \left(\frac{x^2 - 16}{x - 4} \right) = \; ?$$

We deal with indeterminate forms by using methods such as factorization, L'Hôpital's rule, conjugate, amongst other tricks.

Limits - L'Hôpital's rule

Definition

Let's suppose that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0} \text{ or } \frac{\pm\infty}{\pm\infty}$$

where a is a real number and $g'(x) \neq 0,\ \forall x \neq a$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

where f' and g' are the derivatives of f and g, respectively.

Example:

$$\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{\cos(x)}{1} = 1$$

Limits - some additional examples

$$\lim_{x\to 0}\frac{\tan(x)}{x} = ?$$

$$\lim_{x\to 1^+}\frac{x^2-1}{x-1} = ?$$

$$\lim_{x\to 4}\frac{2-\sqrt{x}}{4-x} = ? \text{ (hint use the conjugate: } 2+\sqrt{x}\text{)}$$

$$\lim_{x\to 2}\frac{x^2+4x-12}{x^2-2x} = ? \text{ (hint: factorization)}$$

$$\lim_{x\to 0} x^2\cos\left(\frac{1}{x}\right) = ? \text{ (hint: } -1\leq\cos(x)\leq 1\text{)}$$

$$\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right) = ? \text{ (hint: use conjugate and factorization)}$$

Derivatives

Definition

The derivative of the function f evaluated at a is given by:

$$\left.\frac{df}{dx}\right|_a = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Example:

Computation of the derivative of x^2 at x_0

$$\left.\frac{dx^2}{dx}\right|_{x_0} = \lim_{x \to x_0} \frac{x^2 - x_0^2}{x - x_0} = \lim_{x \to x_0} \frac{(x + x_0)(x - x_0)}{x - x_0} = lim_{x \to x_0}(x + x_0) = 2x_0$$

Derivatives - chain rule

Definition

Composition of functions $f \circ g$ of the variable x, the derivative is as follows:

$$\frac{d}{dx} f \circ g = \left(f' \circ g\right) \times g'$$

Example:

Computation of the derivative of $\sin\left(x^2 + 1\right)$

$$\frac{d}{dx} \sin\left(x^2 + 1\right) = \cos\left(x^2 + 1\right) \times \frac{d}{dx}\left(x^2 + 1\right) = \cos\left(x^2 + 1\right) \times 2x$$

where $f \circ g = \sin\left(x^2 + 1\right)$, $f(u) = \sin(u)$ and $g(x) = x^2 + 1$

Derivatives - example

Find the derivatives of:

$$\begin{aligned}
\frac{d}{dx}\left(e^{x^2+1}\right) &= ? \\
\frac{d}{dx}\tan(x) &= ? \\
\frac{d}{dx}\cos(x) &= ? \\
\frac{d}{dx}\sin(x) &= ? \\
\frac{d}{dx}\ln\left(\cos\left(\frac{1}{x}\right)\right) &= ? \\
\frac{d}{dx}\left(x^4+3x^3+2x^2+1\right) &= ?
\end{aligned}$$

Total differentials and total derivatives

Definition of the total differential

f a function of u and v, $f(u,v)$, the total differential is defined as:

$$df = \left.\frac{\partial f}{\partial u}\right|_{v=cst} du + \left.\frac{\partial f}{\partial v}\right|_{u=cst} dv$$

Definition of the total derivative

f is a function of t, u and v, $f(t,u,v)$ where u and v are also functions of t, the total derivative is defined as:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial u}\frac{du}{dt} + \frac{\partial f}{\partial v}\frac{dv}{dt}$$

Example:

Total differential of $f(u,v) = u^2 + e^v$

$$df = 2u du + e^v dv$$

Integration

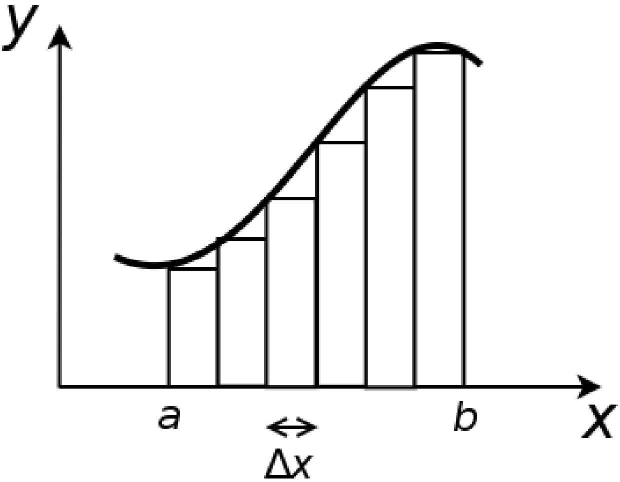
Definition of an integral

An integral represents the sum of a function over an infinitesimal change of the variable. It typically computes surfaces, volumes or averages.
If the function f is continuous over an interval $[a, b]$ divided into n equal segments of length $\Delta x = \dfrac{b-a}{n}$ and $x_{i+1} = x_i + \Delta x$, we can "write" the following definite integral as follows:

$$\int_a^b f(x)\,dx = \lim_{n \to +\infty} \sum_{i=0}^{n-1} f(x_i)\,\Delta x$$

which relates the sum to the integral.

Integration - geometrical interpretation



Integration - different classes of integrals

Riemann and Lebesgue integrals (for your information)

There exist two different kinds of integrals, Riemann and Lebesgue. We are mostly concerned about Riemann's (the usual one). Lebesgue integral extends the concept of integral to ”exotic” functions.

Improper integral

An improper integral is an integral for which the function approaches limits at the endpoints of the interval or the endpoints of the interval includes $\pm\infty$ (concept of convergence).

Example:

$$\int_0^\infty \frac{\sin(x)}{x}dx = \lim_{t\to\infty}\int_0^t \frac{\sin(x)}{x}dx = \frac{\pi}{2}$$

Be careful, not all integrals have a solution in terms of elementary functions.

Integration - Examples

Find the integrals of:

$$\int_0^{\pi} \cos(x)\,dx = \text{? (same for sin)}$$

$$\int_0^{\pi} \cos\left(x^2\right) dx = \text{? (is it integrable?)}$$

$$\int_0^{t} e^{-x} dx = \; , \; t > 0 \text{ ?}$$

$$\int_{-\infty}^{+\infty} e^{-x^2} dx = \text{? (Euler-Poisson or Gaussian integral does have a solution)}$$

$$erf(x) = \int_0^{x} e^{-t^2} dx = \text{? (error function does not have a closed-form solution)}$$

$$\int_0^{+\infty} \frac{1}{x^2} dx = \text{? (improper integral, use limits)}$$

Be careful, not all functions can be integrated. We rely on numerical technique or approximation.

Fundamental theorem of calculus

"Integration and differentiation are inverse operations"

Fundamental theorem of calculus

Let f be a real function, continuous over the interval $[a,b]$. Let F be defined over $[a,b]$ as

$$F(x) = \int_a^x f(x)\,dx$$

then F is continuous on $[a,b]$ and differentiable over the open (a,b). Its derivative is given by:

$$\frac{d}{dx}F(x) = f(x),\ \forall x \in (a,b)$$

Second fundamental theorem of calculus

f is a real function continuous over the interval $[a,b]$ and F is the antiderivative of f in $[a,b]$ then:

$$f(x) = \frac{d}{dx}F(x),\ \forall x \in [a,b]$$

If f is integrable on $[a,b]$, then:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Some bibliography

- MIT OpenCourseWare (videos and material): http://ocw.mit.edu/index.htm
- R. Courant, ”Differential and Integral calculus VI I & II”
- E. Swokowski and J. Abreu, ”Calculo con geometría analítica”, México: Grupo Editorial Iberoamérica, 1989.
- L. Leithold, ”El cálculo, con geometria analítica”, Mexico: Harla S.A. de C.V., 1974.

Thank you for your attention

http://posgrado.electrica.unam.mx

http://ingenieria.posgrado.unam.mx/sitv3/